Lecture 25

Radiation by a Hertzian Dipole

Radiation of electromagnetic field is of ultimate importance for wireless communication systems. The first demonstration of the wave nature of electromagnetic field was by Heinrich Hertz in 1888 [18], some 23 years after Maxwell's equations were fully established. Guglielmo Marconi, after much perserverence with a series of experiments, successfully transmitted wireless radio signal from Cornwall, England to Newfoundland, Canada in 1901 [136]. The experiment was serendipitous since he did not know that the ionosphere was on his side: The ionosphere helped to bounce the radio wave back to earth from outer space. Marconi's success ushered in the age of wireless communication, which is omni-present in our daily lives. Hence, radiation by arbitrary sources is an important topic for antennas and wireless communications. We will start with studying the Hertzian dipole which is the simplest of radiation sources we can think of.

25.1 History

The original historic Hertzian dipole experiment is shown in Figure 25.1. It was done in 1887 by Heinrich Hertz [18]. The schematic for the original experiment is also shown in Figure 25.2.

A metallic sphere has a capacitance in closed form with respect to infinity or a ground plane.[1] Hertz could use those knowledge to estimate the capacitance of the sphere, and also, he could estimate the inductance of the leads that are attached to the dipole, and hence, the resonance frequency of his antenna. The large sphere is needed to have a large capacitance, so that current can be driven through the wires. As we shall see, the radiation strength of the dipole is proportional to $p = ql$ the dipole moment.

[1] We shall learn later that this problem can be solved in closed form using image theorem.

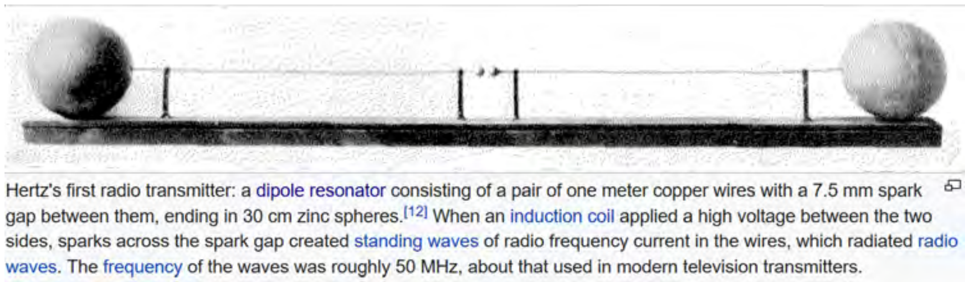

Figure 25.1: Hertz's original experiment on a small dipole (courtesy of Wikipedia [18]).

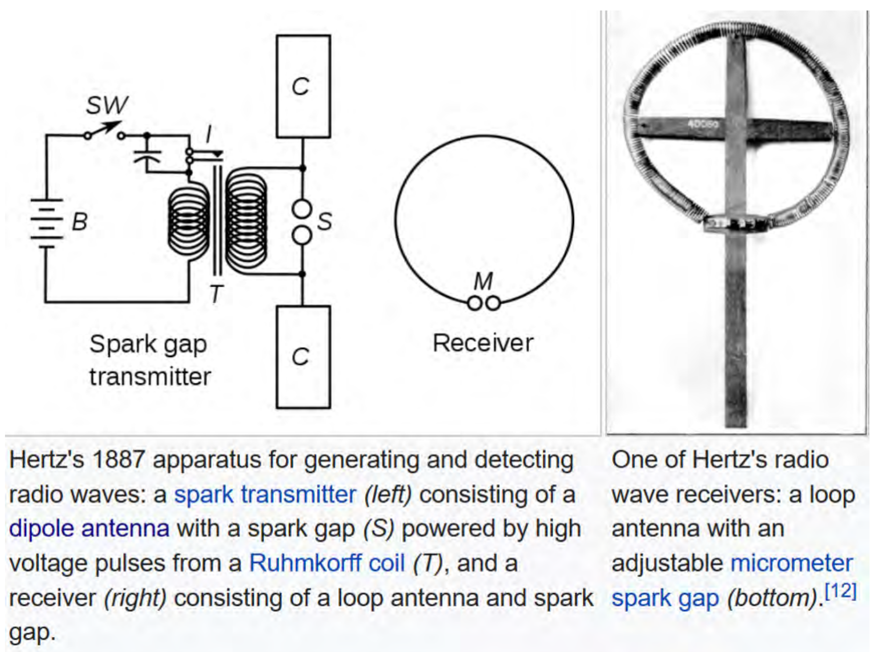

Figure 25.2: More on Hertz's original experiment on a small dipole (courtesy of Wikipedia [18]). The antenna was powered by a transformer. The radiated electromagnetic field was picked up by a loop receiver antenna that generates a spark at its gap M.

25.2 Approximation by a Point Source

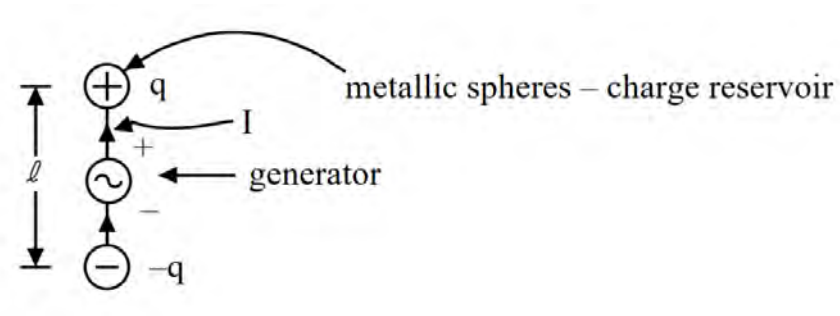

Figure 25.3: Schematic of a small Hertzian dipole which is a close approximation of that first proposed by Hertz.

Figure 25.3 is the schematic of a small Hertzian dipole resembling the original dipole that Hertz made. Assuming that the spheres at the ends store charges of value q, and l is the effective length of the dipole, then the dipole moment $p = ql$. The charge q is varying in time harmonically because it is driven by the generator. Since

$$\frac{dq}{dt} = I,$$

we have the current moment

$$Il = \frac{dq}{dt} l = j\omega ql = j\omega p \tag{25.2.1}$$

for this Hertzian dipole.

A Hertzian dipole is a dipole which is much smaller than the wavelength under consideration so that we can approximate it by a point current distribution, or a current density. Mathematically, it is given by [32, 44]

$$\mathbf{J}(\mathbf{r}) = \hat{z}Il\delta(x)\delta(y)\delta(z) = \hat{z}Il\delta(\mathbf{r}) \tag{25.2.2}$$

The dipole is as shown in Figure 25.3 schematically. As long as we are not too close to the dipole so that it does not look like a point source anymore, the above is a good mathematical model and approximation for describing a Hertzian dipole.

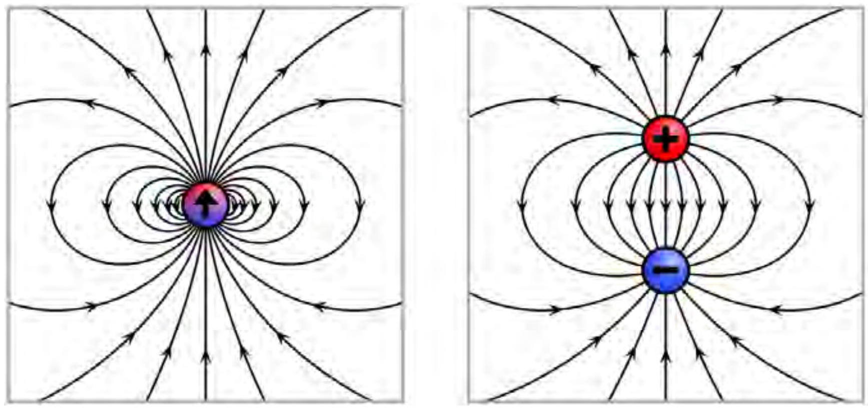

Figure 25.4: The field of a point dipole field versus that of a dipole field. When one is far away from the dipole sources, their fields are similar to each other (courtesy of Wikepedia).

We have learnt previously that the vector potential is related to the current as follows:

$$\mathbf{A}(\mathbf{r}) = \mu \iiint d\mathbf{r}'\mathbf{J}(\mathbf{r}')\frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{25.2.3}$$

Since the current is a 3D delta function in space, using the sifting property of a delta function, the corresponding vector potential is given by

$$\mathbf{A}(\mathbf{r}) = \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r} \tag{25.2.4}$$

Since the vector potential $\mathbf{A}(\mathbf{r})$ is cylindrically symmetric, the corresponding magnetic field is obtained, using cylindrical coordinates, as

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A} = \frac{1}{\mu}\left(\hat{\rho}\frac{1}{\rho}\frac{\partial}{\partial\phi}A_z - \hat{\phi}\frac{\partial}{\partial\rho}A_z\right) \tag{25.2.5}$$

where $\frac{\partial}{\partial\phi} = 0$, $r = \sqrt{\rho^2+z^2}$. In the above, we have used the chain rule that

$$\frac{\partial}{\partial\rho} = \frac{\partial r}{\partial\rho}\frac{\partial}{\partial r} = \frac{\rho}{\sqrt{\rho^2+z^2}}\frac{\partial}{\partial r} = \frac{\rho}{r}\frac{\partial}{\partial r}.$$

As a result,

$$\mathbf{H} = -\hat{\phi}\frac{\rho}{r}\frac{Il}{4\pi}\left(-\frac{1}{r^2} - j\beta\frac{1}{r}\right)e^{-j\beta r} \tag{25.2.6}$$

In spherical coordinates, $\frac{\rho}{r} = \sin\theta$, and (25.2.6) becomes [32]

$$\mathbf{H} = \hat{\phi}\frac{Il}{4\pi r^2}(1+j\beta r)e^{-j\beta r}\sin\theta \tag{25.2.7}$$

The electric field can be derived using Maxwell's equations.

$$\mathbf{E} = \frac{1}{j\omega\epsilon}\nabla\times\mathbf{H} = \frac{1}{j\omega\epsilon}\left(\hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta H_\phi - \hat{\theta}\frac{1}{r}\frac{\partial}{\partial r}rH_\phi\right)$$
$$= \frac{Ile^{-j\beta r}}{j\omega\epsilon 4\pi r^3}\left[\hat{r}2\cos\theta(1+j\beta r) + \hat{\theta}\sin\theta(1+j\beta r-\beta^2r^2)\right] \tag{25.2.8}$$

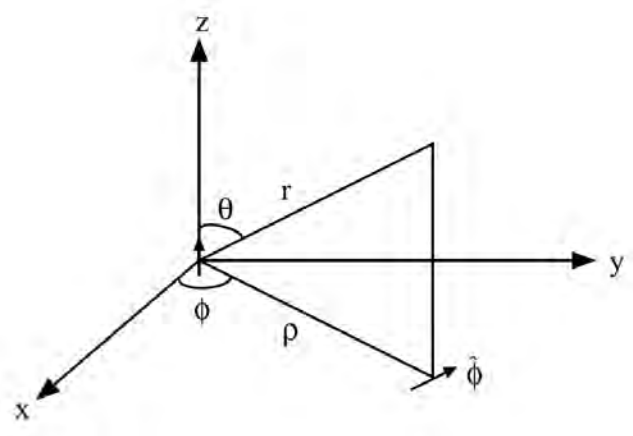

Figure 25.5: Spherical coordinates are used to calculate the fields of a Hertzian dipole.

25.2.1 Case I. Near Field, βr $\ll$ 1

Since $\beta r \ll 1$, retardation effect within this short distance from the point dipole can be ignored. Also, we let $\beta r \to 0$, and keeping the largest terms (or leading order terms in math parlance), then from (25.2.8), with $Il = j\omega p$

$$\mathbf{E} \cong \frac{p}{4\pi\epsilon r^3}(\hat{r}2\cos\theta + \hat{\theta}\sin\theta), \qquad \beta r \ll 1 \tag{25.2.9}$$

For the **H** field, from (25.2.7), with $\beta r \ll 1$, then

$$\mathbf{H} = \hat{\phi}\frac{j\omega p}{4\pi r^2}\sin\theta \tag{25.2.10}$$

or

$$\eta_0\mathbf{H} = \hat{\phi}\frac{j\beta r p}{4\pi\varepsilon r^3}\sin\theta \tag{25.2.11}$$

Thus, it is seen that

$$\eta_0\mathbf{H} \ll \mathbf{E}, \qquad \text{when } \beta r \ll 1 \tag{25.2.12}$$

where $p = ql$ is the dipole moment.[2] The above implies that in the near field, the electric field dominates over the magnetic field.

In the above, βr could be made very small by making $\frac{r}{\lambda}$ small or by making $\omega \to 0$. The above is like the static field of a dipole.

Another viewpoint is that in the near field, the field varies rapidly, and space derivatives are much larger than the time derivative.[3]

For instance,

$$\frac{\partial}{\partial x} \gg \frac{\partial}{c\partial t}$$

Alternatively, we can say that the above is equivalent to

$$\frac{\partial}{\partial x} \gg \frac{\omega}{c}$$

or that

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \approx \nabla^2$$

In other words, static theory prevails over dynamic theory when $\beta r \ll 1$. The above approximations are consistent with that the retardation effect is negligible over this lengthscale.

25.2.2 Case II. Far Field (Radiation Field), $\beta r \gg 1$

This is also known as the far zone. In this case, retardation effect is important. In other words, phase delay cannot be ignored.

$$\mathbf{E} \cong \hat{\theta} j\omega\mu \frac{Il}{4\pi r} e^{-j\beta r} \sin\theta \tag{25.2.13}$$

and

$$\mathbf{H} \cong \hat{\phi} j\beta \frac{Il}{4\pi r} e^{-j\beta r} \sin\theta \tag{25.2.14}$$

Note that $\frac{E_\theta}{H_\phi} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\epsilon}} = \eta_0$. Here, $\mathbf{E}$ and $\mathbf{H}$ are orthogonal to each other and they are both orthogonal to the direction of propagation, as in the case of a plane wave. Or in a word, a spherical wave resembles a plane wave in the far field approximation.

25.3 Radiation, Power, and Directive Gain Patterns

The time average power flow in the far field is given by

$$\langle \mathbf{S} \rangle = \frac{1}{2}\Re e[\mathbf{E} \times \mathbf{H}^*] = \hat{r}\frac{1}{2}\eta_0 |H_\phi|^2 = \hat{r}\frac{\eta_0}{2}\left(\frac{\beta Il}{4\pi r}\right)^2 \sin^2\theta \tag{25.3.1}$$

[2]Here, $\eta_0 = \sqrt{\mu/\varepsilon}$. We multiply $\mathbf{H}$ by η_0 so that the quantities we are comparing have the same unit.

[3]This is in agreement with our observation that electromagnetic fields are great contortionists: They will deform themselves to match the boundary first before satisfying Maxwell's equations. Since the source point is very small, the fields will deform themselves so as to satisfy the boundary conditions near to the source region. If this region is small compared to wavelength, the fields will vary rapidly over a small lengthscale compared to wavelength.

The **radiation field pattern** of a Hertzian dipole is the plot of $|\mathbf{E}|$ as a function of θ at a constant $\mathbf{r}$. Hence, it is proportional to $\sin\theta$, and it can be proved that it is a circle. The **radiation power pattern** is the plot of $\langle S_r \rangle$ at a constant r.

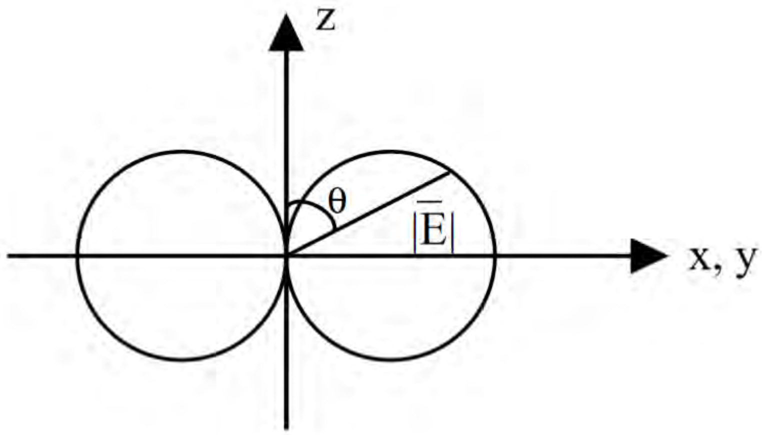

Figure 25.6: Radiation field pattern of a Hertzian dipole. It can be shown that the pattern is a circle.

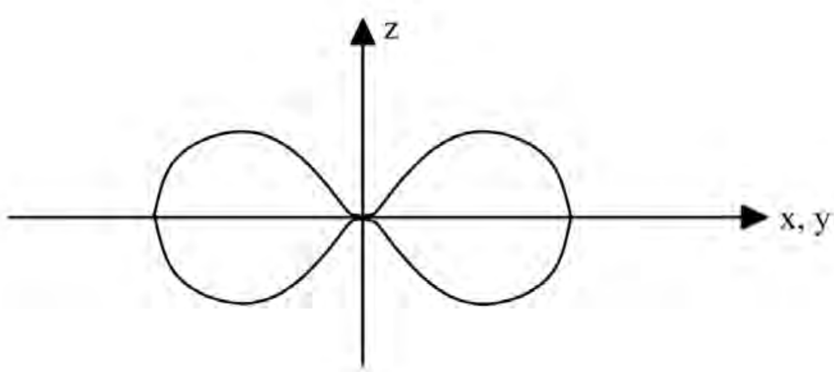

Figure 25.7: Radiation power pattern of a Hertzian dipole which is also the same as the directive gain pattern.

The total power radiated by a Hertzian dipole is thus given by

$$P = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta r^2 \sin\theta \langle S_r \rangle = 2\pi \int_0^{\pi} d\theta \frac{\eta_0}{2} \left(\frac{\beta I l}{4\pi}\right)^2 \sin^3\theta \tag{25.3.2}$$

Since

$$\int_0^{\pi} d\theta \sin^3\theta = -\int_1^{-1} (d\cos\theta)[1-\cos^2\theta] = \int_{-1}^{1} dx(1-x^2) = \frac{4}{3} \tag{25.3.3}$$

then

$$P = \frac{4}{3}\pi\eta_0 \left(\frac{\beta I l}{4\pi}\right)^2 = \frac{\eta_0(\beta I l)^2}{12\pi} \tag{25.3.4}$$

The **directive gain** of an antenna, $G(\theta, \phi)$, is defined as [32]

$$G(\theta, \phi) = \frac{\langle S_r \rangle}{\langle S_{av} \rangle} = \frac{\langle S_r \rangle}{\frac{P}{4\pi r^2}} \tag{25.3.5}$$

where

$$\langle S_{av} \rangle = \frac{P}{4\pi r^2} \tag{25.3.6}$$

is the power density if the power P were uniformly distributed over a sphere of radius r. Notice that $\langle S_{av} \rangle$ is independent of angle. Hence, the angular dependence of the directive gain $G(\theta, \phi)$ is coming from $\langle S_r \rangle$.

Substituting (25.3.1) and (25.3.4) into the above, we have

$$G(\theta, \phi) = \frac{\frac{\eta_0}{2}\left(\frac{\beta I l}{4\pi r}\right)^2 \sin^2\theta}{\frac{1}{4\pi r^2}\frac{4}{3}\eta_0\pi\left(\frac{\beta I l}{4\pi}\right)^2} = \frac{3}{2}\sin^2\theta \tag{25.3.7}$$

The peak of $G(\theta, \phi)$ is known as the **directivity** of an antenna. It is 1.5 in the case of a Hertzian dipole. If an antenna is radiating isotropically, its directivity is 1, which is the lowest possible value, whereas it can be over 100 for some antennas like reflector antennas (see Figure 25.8). A **directive gain pattern** is a plot of the above function $G(\theta, \phi)$ and it resembles the radiation power pattern.

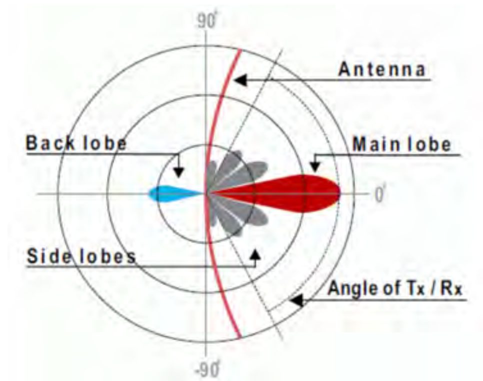

Figure 25.8: The gain of a reflector antenna can be increased by deflecting the power radiated in the desired direction by the use of a reflector (courtesy of racom.eu).

If the total power fed into the antenna instead of the total radiated power is used in the denominator of (25.3.5), the ratio is known as the **power gain** or just **gain** and the pattern is the **power gain pattern**. The total power fed into the antenna is not equal to the total radiated power because there could be some loss in the antenna system like metallic loss.

25.3.1 Radiation Resistance

Engineers love to replace complex systems with simpler systems. Simplicity rules again! This will make interface with electronic driving circuits for the antenna easier to derive. A raw Hertzian dipole, when driven by a voltage source, essentially looks like a capacitor due to the preponderance of electric field energy stored in the dipole field. But at the same time, the dipole radiates giving rise to radiation loss. Thus a simple circuit equivalence of a Hertzian dipole is a capacitor in series with a resistor. The resistor accounts for radiation loss of the dipole.

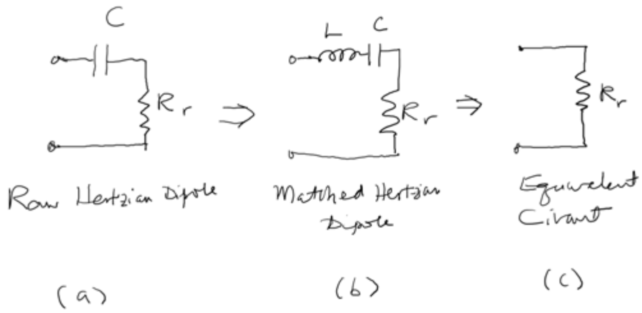

Figure 25.9: (a) Equivalent circuit of a raw Hertzian dipole without matching. (b) Equivalent circuit of a matched Hertzian dipole (using maximum power transfer theorem). (c) Equivalent circuit of a matched dipole at the resonance frequency of the LC tank circuit.

Hence, the way to drive the Hertzian dipole effectively is to use matching network for maximum power transfer. Or an inductor has to be added in series with the intrinsic capacitance of the Hertzian dipole to cancel it at the resonance frequency of the tank circuit. Eventually, after matching, the Hertzian dipole can be modeled as just a resistor. Then the power absorbed by the Hertzian dipole from the driving source is $P = \frac{1}{2}I^2 R_r$. Thus, the **radiation resistance** R_r is the effective resistance that will dissipate the same power as the radiation power P when a current I flows through the resistor. Hence, it is defined by [32]

$$R_r = \frac{2P}{I^2} = \eta_0 \frac{(\beta l)^2}{6\pi} \approx 20(\beta l)^2, \qquad \text{where } \eta_0 = 377 \approx 120\pi\ \Omega \tag{25.3.8}$$

For example, for a Hertzian dipole with $l = 0.1\lambda$, $R_r \approx 8\Omega$.

The above assumes that the current is uniformly distributed over the length of the Hertzian dipole. This is true if there are two charge reservoirs at its two ends. For a small dipole with no charge reservoir at the two ends, the currents have to vanish at the tips of the dipole as shown in Figure 25.10. The effective length of an equivalent Hertzian dipole for the dipole with triangular distribution is **half** of its actual length due to the manner the currents are distributed.[4] Such a formula can be used to estimate the radiation resistance of a dipole.

For example, a half-wave dipole does not have a triangular current distribution a sinusoidal one as shown in Figure 25.11. Nevertheless, we approximate the current distribution of a half-wave dipole with a triangular distribution, and apply the above formula. We pick $a = \frac{\lambda}{2}$, and let $l_{\text{eff}} = \frac{\lambda}{4}$ in (25.3.8), we have

$$R_r \approx 50\Omega \tag{25.3.9}$$

[4]As shall be shown, when the dipole is short, the details of the current distribution is inessential in determining the radiation field. It is the area under the current distribution that is important.

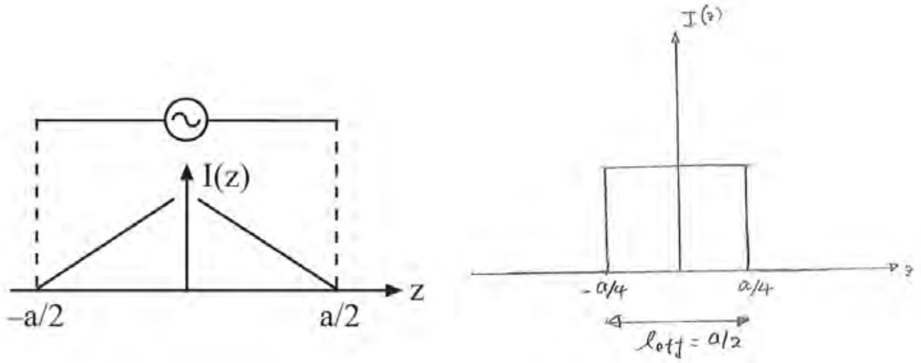

Figure 25.10: The current pattern on a short dipole can be approximated by a triangle since the current has to vanish at the end points of the short dipole. Furthermore, this dipole can be approximated by an effective Hertzian dipole half its length with uniform current.

The true current distribution on a half-wave dipole resembles that shown in Figure 25.11. The current is zero at the end points, but the current has a more sinusoidal-like distribution as in a transmission line. Hence, a half-wave dipole is not much smaller than a wavelength and does not qualify to be a Hertzian dipole. Furthermore, the current distribution on the half-wave dipole is not triangular in shape as above. A more precise calculation shows that $R_r = 73\Omega$ for a half-wave dipole [54]. This also implies that a half-wave dipole with sinusoidal current distribution is a better radiator than a dipole with triangular current distribution.

In fact, one can think of a half-wave dipole as a flared, open transmission line. In the beginning, this flared open transmission line came in the form of biconical antennas which are shown in Figure 25.12 [137]. If we recall that the characteristic impedance of a transmission line is $\sqrt{L/C}$, then as the spacing of the two metal pieces becomes bigger, the equivalent characteristic impedance gets bigger. Therefore, the impedance can gradually transform from a small impedance like 50 Ω to that of free space, which is 377 Ω. This impedance matching helps mitigate reflection from the ends of the flared transmission line, and enhances radiation. Because of the matching nature of bicone antennas, they are better radiators with higher radiaton loss and lower Q. Thus they have a broader bandwidth, and are important in UWB (ultra-wide band) antennas [138].

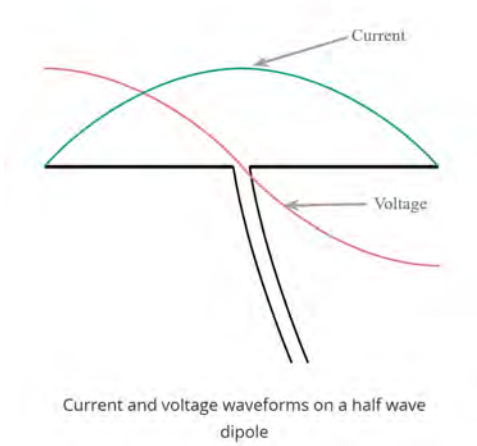

Figure 25.11: Approximate current distribution on a half-wave dipole (courtesy of electronics-notes.co). The currents are zero at the two end tips due to the current continuity equation, or KCL.

Figure 25.12: A bicone antenna can be thought of as a transmission line with gradually changing characteristic impedance. This enhances impedance matching and the radiation of the antenna (courtesy of antennasproduct.com).